Pass-Band VSWR of Maximally Flat Band-Pass Filters*

Microwave band-pass filters using high-Q cavity resonators can often be designed for almost negligible midband dissipation losses (*i.e.*, $<\frac{1}{2}$ db). If these filters are designed for a maximally flat amplitude response, the insertion loss within the pass band is approximately equal to the reflection loss within the pass band. Then

$$R = 10 \log (1 + X^{2n}),$$

where

R = reflection loss in db
X = normalized frequency variable
n = number of cavity resonators.

Letting ρ = input VSWR,

$$\rho = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + X^n}{1 - X^n}.$$

Curves as ρ vs X have been plotted in Fig. 1 using the above equation for values of n from 1 to 8.

If the approximation of $X^{2n} \ll 1$ is not used, it can be shown that

$$\rho = \frac{2 + 4X^{2n} + \sqrt{(2 + 4X^{2n})^2 - 4}}{2}.$$

If $X^{2n} = 0.125$, $\rho = 2.0$ using the exact equation for ρ. If the approximate equation is used, with $X^{2n} = 0.125$ ρ is equal to 2.1. It can be concluded that the curves shown are satisfactory for most purposes when $\rho \leqq 2.0$.

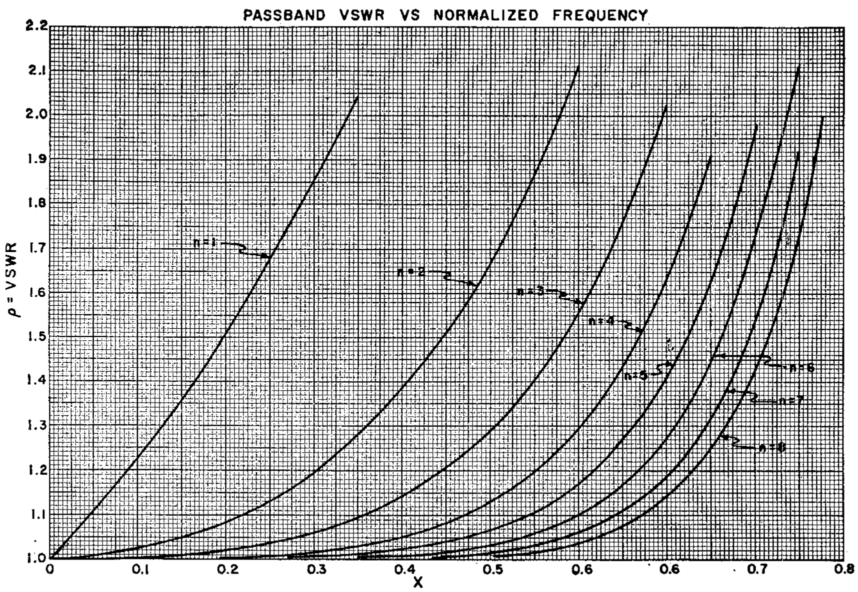

Fig. 1.

For narrow-band filters,

$$X \cong 2 \left| \frac{f - f_0}{\Delta f} \right|,$$

where

f = frequency
f_0 = center frequency of filter
Δf = 3-db bandwidth of filter.

Now

$$1 + X^{2n} = \frac{1}{1 - |\Gamma|^2},$$

where

Γ = voltage reflection coefficient looking into the filter

$$1 - |\Gamma|^2 = \frac{1}{1 + X^{2n}} \cong 1 - X^{2n} \quad \text{if} \quad X^{2n} \ll 1$$

$$|\Gamma|^2 = X^{2n}$$

$$|\Gamma| = X^n.$$

The approximation used can also be applied to deriving equations for the VSWR of quasi-dissipationless filters of other response shapes, such as constant $-K$ or Tchebycheff.

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* Received by the PGMTT, December 12, 1961.

An Approximate Method of Finding the Order of a Combination Bessel-Function Equation*

Assume a transverse electric wave, TE mode, propagated in a waveguide with a circular boundary. The radial boundary condition gives a combination Bessel function equation; *i.e.*,

$$Z_p'(x_i) = J_p'(x)N_p'(x_i) - N_p'(x)J_p'(x_i) = 0, \tag{1}$$

where

$$x = \beta_c \rho_o \qquad x_i = \beta_c \rho_i.$$

The terms ρ_o and ρ_i are the radius of the circular boundaries of the waveguide, β_c is the cutoff wave number, and p is the order of the Bessel functions.

For a coaxial waveguide, p is determined from the angular boundary and is an integer such as 0, 1, 2, etc.

For a lunar line,[1] β_c is assumed and substituted in (1) to solve p. The arguments X and X_i are small, and the series form of the Bessel functions converges rapidly. By taking finite terms and using a graphic method, two real roots of p are obtained: one positive and one equal negative root. It is very complicated and difficult to find the complex roots using this method. However, using the following approximate method, the real roots and the complex roots p of (1) can be obtained. Eq. (1) can be written as

$$[J_p'(x_i)/J_p'(x)] = [N_p'(x_i)/N_p'(x)], \tag{2}$$

where

$$0 < (x_i - x) < 1.$$

By the Taylor series expansion,

$$J_p'(x_i) = J_p'(x) + (x_i - x)J_p''(x) + \frac{(x_i - x)^2}{2!} J_p'''(x) + \cdots. \tag{3}$$

Substituting the series forms of $J_p'(x_i)$ and $N_p'(x_i)$ into (2) simplifies that equation to

$$[J_p''(x)N_p'(x) - N_p''(x)J_p'(x)] + \frac{(x_i - x)}{2!}[J_p'''(x)N_p'(x) - N_p'''(x)J_p'(x)] + \frac{(x_i - x)^2}{3!}[J_p^{IV}(x)N_p'(x) - N_p^{IV}(x)J_p'(x)] + \frac{(x_i - x)^3}{4!}[J_p^{V}(x)N_p'(x) - N_p^{V}(x)J_p'(x)] + \cdots = 0. \tag{4}$$

From the differential equations,

$$J_p''(x) + \frac{1}{x}J_p'(x) + \left(1 - \frac{p^2}{x^2}\right)J_p(x) = 0 \tag{5}$$

and

$$N_p''(x) + \frac{1}{x}N_p'(x) + \left(1 - \frac{p^2}{x^2}\right)N_p(x) = 0. \tag{6}$$

The difference of (5) times $N_p'(x)$ and (6) times $J_p'(x)$ gives

$$J_p''(x)N_p'(x) - N_p''(x)J_p'(x) = \frac{2p^2}{\pi x^3} - \frac{2}{\pi x}. \tag{7}$$

* Received by the PGMTT, December 7, 1961.

[1] A. Y. Hu and A. Ishimaru, "The dominant cut-off wavelength of a lunar line," IRE Trans. on Microwave Theory and Techniques, vol. MTT-9, pp. 552–556; November, 1961.

The difference of differential (5) times $N_p'(x)$ and differential (6) times $J_p'(x)$ gives

$$J_p'''(x)N_p'(x) - N_p'''(x)J_p'(x) = -\frac{6p^2}{\pi x^4} + \frac{2}{\pi x^2}. \quad (8)$$

Similarly

$$J_p^{IV}(x)N_p'(x) - N_p^{IV}(x)J_p'(x) = \frac{2p^4}{\pi x^5} + \frac{22p^2}{\pi x^5} - \frac{4p^2}{\pi x^3} - \frac{6}{\pi x^3} + \frac{2}{\pi x} \quad (9)$$

$$J_p^{V}(x)N_p'(x) - N_p^{V}(x)J_p'(x) = -\frac{20p^4}{\pi x^6} - \frac{100p^2}{\pi x^6} + \frac{24p^2}{\pi x^4} + \frac{24}{\pi x^4} - \frac{4}{\pi x^2} \cdots \text{etc.} \quad (10)$$

If $(x_i - x)$ is very small, the higher power terms of $(x_i - x)$ may be neglected for the approximate computation. Substituting (7)–(9) into (4) and dropping the terms higher than $(x_i - x)^2$ yields

$$(x_i - x)^2p^4 + (11x_i^2 - 31x_ix + 26x^2 - 2x_i^2x^2 + 4x_ix^3 - 2x^3)p^2 + (x^6 - 2x_ix^5 + x_i^2x^4 - 3x_i^2x^2 + 9x_ix^3 - 12x^4) = 0. \quad (11)$$

Substituting (7)–(10) into (4) and dropping the terms higher than $(x_i - x)^3$ gives

$$p^4\left[(x_i - x)^2 - \frac{5(x_i - x)^3}{2x}\right] + p^2\left\{6x^2 - 9x(x_i - x) + (11 - 2x^2)(x_i - x)^2 + (x_i - x)^3\left[-\frac{25}{2x} + 3x\right]\right\} + \left\{3x^3(x_i - x) + (x^4 - 3x^2)(x_i - x)^2 - 6x^4 + (x_i - x)^3\left[3x - \frac{x^3}{2}\right]\right\} = 0. \quad (12)$$

Sample of calculation: if $x_i = 0.6566$ and $x = 0.544375$, substituting x_i and x into (11) yields

$$p = \pm 0.59495 \quad \text{and} \quad p = \pm j10.406;$$

substituting x_i and x into (12) yields

$$p = \pm\ 0.60078 \quad \text{and} \quad p = \pm j14.769.$$

The real roots computed from the Bessel function series expansion are ± 0.599 and are close to the values obtained from (11) and (12). The first pairs of complex roots obtained with (11) are different from those obtained with (12). The complex roots of (4) must be solved with more higher-power terms to obtain a close value. However, these complex roots are large and the field will attenuate quickly.

ACKNOWLEDGMENT

The author wishes to express her appreciation to Prof. Bernard Friedman, Department of Mathematics, University of California, for his valuable suggestion.

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A Source of Error in the Use of Slope Detection for Perturbation Measurements*

In a typical laboratory measurement of the field-intensity distribution in a resonant cavity, the resonant frequency of the cavity is determined as a function of the position of a small metal or dielectric bead. The field distribution itself is then deduced directly from the frequency change.

A simple scheme advanced by Ayers, Chu and Gallagher[1] depends on the fact that a graph of the logarithm of the response of a cavity vs frequency is very nearly linear between 1 and 7 db below the maximum response. In this region, the frequency perturbation caused by a bead is directly proportional to the change of attenuation through the cavity with a maximum error of ± 1 per cent of the range. A fixed-frequency source is tuned slightly off the unperturbed resonance of the cavity; any of a number of circuits for measuring the change of insertion loss of the cavity may be used; and a large amount of data may be taken quickly.

This scheme of "slope detection" has one disadvantage: the power range in which the variation is linear is only 6 db, and high precision is difficult to attain. It would appear reasonable to relinquish the linear property and to use larger perturbations which reduce the signal transmissions through the cavity as much as 20 db. It would be required, of course, that the Q and coupling to the cavity remain unchanged.

It is the purpose of this note to demonstrate that the effective coupling is changed, however, if the frequency perturbation is much larger than the unperturbed bandwidth of the cavity. The phase shift introduced by the bead can cause the excitation at one end of the cavity to be greater than that at the other so that, for a given excitation at the vicinity of the input coupling loop, the total stored energy in the cavity may vary. Thus, the coupling efficiency of the loop and the transmission efficiency of the cavity no longer follow the first-order predictions.

Consider the cavity of Fig. 1, consisting of a section of uniform transmssion line of length L symmetrically coupled by ideal transformers of turns-ratio n arranged to produce a voltage maximum at each end of the cavity. The propagation constant at resonance is $\gamma = \alpha + j\beta$. A small change of the operating frequency will produce little change of the attenuation constant α but will change the phase constant to a new value $\beta + \epsilon$, where ϵ is proportional to the frequency change. A small perturbing bead with normalized admittance jB is introduced at a distance l from one end of the cavity. The effect of the bead may be described by computing the over-all transmission coefficient S_{12} of the circuit.[2]

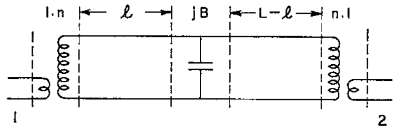

Fig. 1—Equivalent circuit of cavity.

For the usual case of a weakly coupled cavity, one finds, to first-order in $\Delta = \alpha l$,

$$|S_{12}|^2 = \frac{k(1 - \Delta/2)}{1 + (\delta + b\cos^2\beta l)^2 + \frac{1}{2}\Delta b^2\cos^3\beta l[4(1 - 2l/L)\sin\beta l - \cos\beta l]}, \quad (1)$$

where $k = 1/n^2\alpha L$ is the coupling coefficient, $b = B/\alpha L$ is a renormalization of the bead susceptance, Δ is a small quantity which represents the loss within the cavity, and δ is the frequency shift, normalized to the half-bandwidth of the cavity. Note that $\cos \alpha l$ represents the unperturbed electric field distribution.

It is the last term in the denominator which leads to the distortion of the response. It appears to be small since it is of order, Δ, but the term is also proportional to b^2. From the first term, it will be recognized that b is the ratio of the maximum frequency perturbation introduced by the bead to the half-bandwidth of the cavity. If the bead is large enough to shift the resonant frequency several bandwidths, the factor b^2 may well be large enough to make the factor in $1 - 2l/L$ significant. (It should be noted that, since the frequency variable δ does not enter this term, the resonant frequency itself continues to follow the first-order perturbation theory.)

This effect has been observed in tests of $2\pi/3$ mode disk-loaded accelerator sections. For a 3-disk cavity, one wavelength long, a frequency perturbation curve of the form of Fig. 2(a) was observed. The data were obtained using slope detection and a large bead which introduced a maximum change of transmission of -19 db. The curve appeared to agree with expectations except for the end effect where the bead interacts with its image in the end-plate. To eliminate the end-plate errors, a second measurement was made with a 6-disk (two-wavelength) cavity. It was expected that the curve would be identical to two of the original curves pasted together except, of course, for the elimination of end-plate effects on the central hump. Instead, the curve was found to be considerably distorted, as shown in Fig. 2(b).

In the latter cavity the voltage attenuation was relatively large, so that $\Delta = \alpha L$

* Received by the PGMTT, June 12, 1961; revised manuscript received, December 2, 1961. This work was supported by the U. S. Atomic Energy Commission.

[1] W. R. Ayers, E. L. Chu, and W. J. Gallagher, "Measurements of Interaction Impedance in Periodic Circuits," W. W. Hansen Labs. of Physics, Stanford University, Stanford, Calif., Internal Memo., ML Rept. No. 403; June, 1957.

[2] This analysis is presented without regard for space-harmonics in the cavity. The author has shown in a recent paper, "A Perturbation Technique for Impedance Measurements" (presented at the IEE Conf. on Microwave Measurements and Techniques, London, Eng., September 6, 1961), that the presence of space harmonics modifies the effective susceptance of the bead in a manner strictly periodic with the loading elements of the structure.